Duality as a fundamental component of physics

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1 Mathematical duality

At the heart of both physics and mathematics is a fundamental principle of duality. Essentially, this is a way of creating 'something from nothing'. If the ultimate thing that we wish to describe is really 'nothing', then we can only create 'something' as part of a dual pair, in which each thing is opposed by another thing which negates it. A simple physical example occurs in the conservation of momentum, where a system such as a gun and bullet has the same zero momentum after firing that it had before, because gun and bullet acquire equal momentum in opposite directions. We can describe this mathematically in terms of the simplest know symmetry group (C_2), essentially equivalent to an object and its mirror image (or 'dual'), whose components are the positive and negative versions of a quantity which may be left undefined. Duality, however, is not a single operation, and the process requires indefinite extension, in the form $C_2 \times C_2 \times C_2 \times \dots$.

Suppose we begin with a unit (1) and then imagine finding an infinite series of 'duals' to this unit. Let us suppose also that the dualling process must be carried out with respect to all previous duals (that is, that the entire set of characters generated becomes the new 'unit') and that its total result is zero at every stage. Then, the first dual will clearly be –1 (with the first new 'unit' becoming 1, –1), and the next a series of terms to which we could give symbols such as $\boldsymbol{i}_1$, $\boldsymbol{j}_1$, etc.

order 2	$(1, -1)$
order 4	$(1, -1) \times (1, \boldsymbol{i}_1)$
order 8	$(1, -1) \times (1, \boldsymbol{i}_1) \times (1, \boldsymbol{j}_1)$
order 16	$(1, -1) \times (1, \boldsymbol{i}_1) \times (1, \boldsymbol{j}_1) \times (1, \boldsymbol{i}_2)$
order 32	$(1, -1) \times (1, \boldsymbol{i}_1) \times (1, \boldsymbol{j}_1) \times (1, \boldsymbol{i}_2) \times (1, \boldsymbol{j}_2)$
order 64	$(1, -1) \times (1, \boldsymbol{i}_1) \times (1, \boldsymbol{j}_1) \times (1, \boldsymbol{i}_2) \times (1, \boldsymbol{j}_2) \times (1, \boldsymbol{i}_3)$, etc.

The new 'units' are the entire sets of characters generated in this way. That is, at order 2 we have a 'unit' with 2 characters; at order 4, we have a 'unit' with 4 characters; at order 8, a 'unit' with 8 characters, and so on. Writing them out in full, we obtain:

order 2	± 1
order 4	$\pm 1, \pm \boldsymbol{i}_1$
order 8	$\pm 1, \pm \boldsymbol{i}_1, \pm \boldsymbol{j}_1, \pm \boldsymbol{i}_1\boldsymbol{j}_1$
order 16	$\pm 1, \pm \boldsymbol{i}_1, \pm \boldsymbol{j}_1, \pm \boldsymbol{i}_1\boldsymbol{j}_1, \pm \boldsymbol{i}_2, \pm \boldsymbol{i}_2\boldsymbol{i}_1, \pm \boldsymbol{i}_2\boldsymbol{j}_1, \pm \boldsymbol{i}_2\boldsymbol{i}_1\boldsymbol{j}_1$
order 32	$\pm 1, \pm \boldsymbol{i}_1, \pm \boldsymbol{j}_1, \pm \boldsymbol{i}_1\boldsymbol{j}_1, \pm \boldsymbol{i}_2, \pm \boldsymbol{i}_2\boldsymbol{i}_1, \pm \boldsymbol{i}_2\boldsymbol{j}_1, \pm \boldsymbol{i}_2\boldsymbol{i}_1\boldsymbol{j}_1,$
	$\pm \boldsymbol{j}_2, \pm \boldsymbol{j}_2\boldsymbol{i}_1, \pm \boldsymbol{j}_2\boldsymbol{j}_1, \pm \boldsymbol{j}_2\boldsymbol{i}_1\boldsymbol{j}_1, \pm \boldsymbol{j}_2\boldsymbol{i}_2, \pm \boldsymbol{j}_2\boldsymbol{i}_2\boldsymbol{i}_1, \pm \boldsymbol{j}_2\boldsymbol{i}_2\boldsymbol{j}_1, \pm \boldsymbol{j}_2\boldsymbol{i}_2\boldsymbol{i}_1\boldsymbol{j}_1$
order 64	$\pm 1, \pm \boldsymbol{i}_1, \pm \boldsymbol{j}_1, \pm \boldsymbol{i}_1\boldsymbol{j}_1, \pm \boldsymbol{i}_2\boldsymbol{i}_1, \pm \boldsymbol{i}_2\boldsymbol{i}_1, \pm \boldsymbol{i}_2\boldsymbol{j}_1, \pm \boldsymbol{i}_2\boldsymbol{i}_1\boldsymbol{j}_1,$
	$\pm \boldsymbol{j}_2, \pm \boldsymbol{j}_2\boldsymbol{i}_1, \pm \boldsymbol{j}_2\boldsymbol{j}_1, \pm \boldsymbol{j}_2\boldsymbol{i}_1\boldsymbol{j}_1, \pm \boldsymbol{j}_2\boldsymbol{i}_2, \pm \boldsymbol{j}_2\boldsymbol{i}_2\boldsymbol{i}_1, \pm \boldsymbol{j}_2\boldsymbol{i}_2\boldsymbol{j}_1, \pm \boldsymbol{j}_2\boldsymbol{i}_2\boldsymbol{i}_1\boldsymbol{j}_1$
	$\pm \boldsymbol{i}_3, \pm \boldsymbol{i}_3\boldsymbol{i}_1, \pm \boldsymbol{i}_3\boldsymbol{j}_1, \pm \boldsymbol{i}_3\boldsymbol{i}_1\boldsymbol{j}_1, \pm \boldsymbol{i}_3\boldsymbol{i}_2, \pm \boldsymbol{i}_3\boldsymbol{i}_2\boldsymbol{i}_1, \pm \boldsymbol{i}_3\boldsymbol{i}_2\boldsymbol{j}_1, \pm \boldsymbol{i}_3\boldsymbol{i}_2\boldsymbol{i}_1\boldsymbol{j}_1,$
	$\pm \boldsymbol{i}_3\boldsymbol{j}_2, \pm \boldsymbol{i}_3\boldsymbol{j}_2\boldsymbol{i}_1, \pm \boldsymbol{i}_3\boldsymbol{j}_2\boldsymbol{j}_1, \pm \boldsymbol{i}_3\boldsymbol{j}_2\boldsymbol{i}_1\boldsymbol{j}_1, \pm \boldsymbol{i}_3\boldsymbol{j}_2\boldsymbol{i}_2, \pm \boldsymbol{i}_3\boldsymbol{j}_2\boldsymbol{i}_2\boldsymbol{i}_1, \pm \boldsymbol{i}_3\boldsymbol{j}_2\boldsymbol{i}_2\boldsymbol{j}_1, \pm \boldsymbol{i}_3\boldsymbol{j}_2\boldsymbol{i}_2\boldsymbol{i}_1\boldsymbol{j}_1$

For these sets of characters to be true ‘units’, we need to ensure that the products of any unit with itself, or with any subunit, generates only the unit. For example, at order 8, we will have the products:

$$(\pm 1) \times (\pm 1, \pm \boldsymbol{i}_1, \pm \boldsymbol{j}_1, \pm \boldsymbol{i}_1\boldsymbol{j}_1) = (\pm 1, \pm \boldsymbol{i}_1, \pm \boldsymbol{j}_1, \pm \boldsymbol{i}_1\boldsymbol{j}_1)$$
$$(\pm \boldsymbol{i}_1) \times (\pm 1, \pm \boldsymbol{i}_1, \pm \boldsymbol{j}_1, \pm \boldsymbol{i}_1\boldsymbol{j}_1) = (\pm 1, \pm \boldsymbol{i}_1, \pm \boldsymbol{j}_1, \pm \boldsymbol{i}_1\boldsymbol{j}_1)$$
$$(\pm 1, \pm \boldsymbol{i}_1) \times (\pm 1, \pm \boldsymbol{i}_1, \pm \boldsymbol{j}_1, \pm \boldsymbol{i}_1\boldsymbol{j}_1) = (\pm 1, \pm \boldsymbol{i}_1, \pm \boldsymbol{j}_1, \pm \boldsymbol{i}_1\boldsymbol{j}_1)$$
$$(\pm 1, \pm \boldsymbol{i}_1, \pm \boldsymbol{j}_1, \pm \boldsymbol{i}_1\boldsymbol{j}_1) \times (\pm 1, \pm \boldsymbol{i}_1, \pm \boldsymbol{j}_1, \pm \boldsymbol{i}_1\boldsymbol{j}_1) = (\pm 1, \pm \boldsymbol{i}_1, \pm \boldsymbol{j}_1, \pm \boldsymbol{i}_1\boldsymbol{j}_1), \text{ etc.}$$

To ensure that this is true in all case, and that the total sum remains zero, we are constrained into making the terms $\boldsymbol{i}_1$, $\boldsymbol{j}_1$, $\boldsymbol{i}_2$, $\boldsymbol{j}_2$, $\boldsymbol{i}_3$, $\boldsymbol{j}_3$, etc. into *complex numbers*, or square roots of –1, while the products, such as $\boldsymbol{i}_1\boldsymbol{j}_1$, must be complex or real units, that is, square roots of either –1 or +1. The choice here is, in principle, arbitrary, but restricted. That is, we can generate an unlimited number of complex products which are square roots of 1, but, for any complex number, such as $\boldsymbol{i}_1$, there is *only a single complex product* of the form $\boldsymbol{i}_1\boldsymbol{j}_1$, which is itself complex. That is, if $\boldsymbol{i}_1\boldsymbol{j}_1$ is complex, then $\boldsymbol{i}_1$, $\boldsymbol{j}_1$, and $\boldsymbol{i}_1\boldsymbol{j}_1$ form a *closed system*: $\boldsymbol{i}_1$ has no complex product with any other complex number.

Because of its special nature the system $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{ij}$, where $\boldsymbol{ij}$ is also complex, defines a natural concept of 3-dimensionality. If we express $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{ij}$, in the form $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$, we recognize that it has the structure of the cyclic *quaternion* system of complex numbers. The system is closed with the dimensionality fixed at 3, and no other option is available. Remarkably, then, dimensionality becomes simply a manifestation of duality. If $\boldsymbol{i}_1$ forms a quaternion system with $\boldsymbol{j}_1$, and the complex product $\boldsymbol{i}_1\boldsymbol{j}_1$, then no product of $\boldsymbol{i}_1$ with any other complex number of the form $\boldsymbol{i}_1$, $\boldsymbol{i}_2$, $\boldsymbol{i}_3$, $\boldsymbol{j}_4$, … or $\boldsymbol{j}_2$, $\boldsymbol{j}_3$, $\boldsymbol{j}_3$, … will itself be complex. As a result of this, we can define two separate processes of dualling through complex numbers: the ordinary process of complexification (i.e. multiplying by a complex number), which can be continued to infinity (here symbolized by using terms of the form $\boldsymbol{i}_n$); and the restricted process of dimensionalization (here symbolized by using additional terms of the form $\boldsymbol{j}_n$ for each $\boldsymbol{i}_n$), which applies separately, and uniquely, to every complex operator.

We can, therefore, imagine the dualling process taking place through the production of an infinite number of quaternion systems, with the first stage of each representing an ordinary complexification. In addition to the ordinary quaternion operators, therefore, which follow the multiplication rules:

$$\begin{gathered}\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1 \\ \boldsymbol{ij} = -\boldsymbol{ji} = \boldsymbol{k} \\ \boldsymbol{jk} = -\boldsymbol{kj} = \boldsymbol{i} \\ \boldsymbol{ki} = -\boldsymbol{ik} = \boldsymbol{j} \\ \boldsymbol{ijk} = -1 \,,\end{gathered}$$

we have those for complexified quaternions, which are:

$$\begin{gathered}(i\boldsymbol{i})^2 = (i\boldsymbol{j})^2 = (i\boldsymbol{k})^2 = 1 \\ (i\boldsymbol{i})(i\boldsymbol{j}) = -(i\boldsymbol{j})(i\boldsymbol{i}) = i(i\boldsymbol{k}) \\ (i\boldsymbol{j})(i\boldsymbol{k}) = -(i\boldsymbol{k})(i\boldsymbol{j}) = i(i\boldsymbol{i}) \\ (i\boldsymbol{k})(i\boldsymbol{i}) = -(i\boldsymbol{i})(i\boldsymbol{k}) = i(i\boldsymbol{j}) \\ (i\boldsymbol{i})(i\boldsymbol{j})(i\boldsymbol{k}) = i \,,\end{gathered}$$

In principle, these are the objects which we define as multivariate *vectors*, **i**, **j**, **k**, with the rules:

$$\begin{gathered}\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = 1 \\ \mathbf{ij} = -\mathbf{ji} = i\mathbf{k} \\ \mathbf{jk} = -\mathbf{kj} = i\mathbf{i} \\ \mathbf{ki} = -\mathbf{ik} = i\mathbf{j} \\ \mathbf{ijk} = i \,.\end{gathered}$$

It is significant that all these operators are *anticommutative*, with, for example, $\boldsymbol{i}_1\boldsymbol{j}_1 = -\boldsymbol{j}_1\boldsymbol{i}_1$. This is a property which applies strictly to 3-dimensional quantities; operations such as $\boldsymbol{i}_1\boldsymbol{i}_2$ are *commutative*, with $\boldsymbol{i}_1\boldsymbol{i}_2 = \boldsymbol{i}_2\boldsymbol{i}_1$.

Mathematically, then, our dualling process leads to a whole succession of stages, from positive and negative integers, to complex numbers, quaternions (or ‘dimensional’ complex numbers), complex quaternions (or vectors), double quaternions, complex double (or vector) quaternions, and so on. Each stage is derived as the simplest possible way of providing a dual for the previous stage. However, after the quaternion stage, the process becomes repetitive, and so there appear to be only three independent ways in which a dual may be found: conjugation (+ and –), complexification (introducing complex numbers alongside real ones), and dimensionalization (making the numbers 3-

dimensional). We can generate a whole mathematical structure, with a seemingly unlimited power of application, simply from the assumption of duality. Clearly, duality is fundamental to the whole concept.

order 2	real scalar
order 4	complex scalar (pseudoscalar)
order 8	quaternions
order 16	complex quaternions or multivariate vectors
order 32	double quaternions
order 64	complex double quaternions or multivariate vector quaternions, etc.

2 Physical duality

The inherent duality in mathematics seems also to be reflected in the concept of duality in physics. It appears also that the entire structure of physics can be derived from defining fundamental parameters mass, time, charge, and space, which incorporate the dual processes in an exactly symmetrical structure (equivalent to $C_2 \times C_2$). This structure has been described previously in the paper 'Why does physics work?'[1] It can be summarised in the form:

space	nonconserved	real	dimensional / discrete
time	nonconserved	imaginary	nondimensional / continuous
mass	conserved	real	nondimensional /continuous
charge	conserved	imaginary	dimensional / discrete

or, alternatively, using the mathematical dualling processes described in the previous section:

space	nonconjugated	real	dimensional
time	nonconjugated	complex	nondimensional
mass	conjugated	real	nondimensional
charge	conjugated	complex	dimensional

That this symmetry is exact and absolute has been argued in previous papers, where applications have been made to many aspects of physics.[2-5] The exactness of the symmetry, with a perfect distribution between properties and 'antiproperties', creates the possibility of a total conceptual nothingness. In general terms, physics structures itself by defining systems in which conserved quantities remain fixed while nonconserved quantities vary absolutely. Here, conserved quantities are also 'conjugated', meaning that positive values of, say, charge or units of mass-energy cannot be created or destroyed without the simultaneous creation or destruction of negative ones. (This also applies separately to the three types of charge which make up its ‘dimensions’: electric, strong and weak.) Dimensional quantities are also firmly identified with discrete ones. One cannot imagine, for example, a mechanism for dividing the units in a single dimension. It is because this is impossible for time that it becomes physically irreversible, and for the same reason mass-energy becomes physically unipolar (with only one sign, and profound consequences for quantum mechanics and particle physics) – neither quantity allows a dimensional discontinuity or origin representing a zero state.

The parameter group may be thought of in terms of three separate dualities, and these have many physical manifestations. The first duality (conserved / nonconserved, conjugated / nonconjugated) manifests itself in the use of pairs of *conjugate variables* to define a system, in both classical and in quantum physics. In each case, a conserved quantity is paired with a nonconserved one. So momentum is paired with space, and energy with time. The (vector) momentum is said to constitute an alternative *phase space*, and the techniques of Fourier analysis can be used to transform a representation in real space into one in phase space and vice versa. In quantum mechanics, the conjugate variables are the ones limited by the Heisenberg uncertainty relations: too much precision in one leads to a loss of precision in the other.

$$(\Delta p)\,(\Delta x) \geq \hbar\ /\ 2 \quad ; \qquad (\Delta E)\,(\Delta t) \geq \hbar\ /\ 2 \quad .$$

In the concept of ‘virtual’ particles, we are able to consider states in which we can temporarily ‘exchange’ information about space for information about momentum, and information about time for information about energy. Now, the parameter group suggests that the *fundamental* conjugation of space should be with charge rather than momentum, and that the *fundamental* conjugation of time should be with mass rather than energy, but particle physics shows that there are deep connections between the concepts of charge and momentum (or angular momentum), and the relativistic connection between mass and energy has long been established.

The real / complex duality is the relativistic one. This allows transformations to be made, for example, between space and time representations; and it also manifests itself in the well-known duality between electric and magnetic fields which is apparent in Maxwell’s equations. A more subtle form of it occurs in the creation of massive particle states at the expense of components of charge. The third duality can be represented in terms of the discrete / continuous, or the dimensional / nondimensional options. A classic case of the first representation is the well-known wave-particle duality, where waves represent continuous options and particles discrete ones. The reason why it is an option is that the theories apply continuity or discreteness to the *entire system,* instead of only to those components which are fundamentally continuous or discrete; and the balance has to be restored by allowing the possibility of the alternative option. The same applies to the

Heisenberg and Schrödinger theories of quantum mechanics, which are, respectively, discrete and continuous, but which each have to incorporate some aspect of the excluded property when applied to a real physical system.

3 Visual representations

The multiple duality of the parameter group has many strikingly simple visual representations using colours and geometric diagrams in 3 dimensions. These become possible because dimensionality is a direct result of duality, and a 3-colour system is a way of simulating 3-dimensionality. For example, the four parameters, space, time, mass and charge, are represented by concentric circles, the parameter arbitrarily chosen as the identity element for the group occupying the centre circle, while a division of the circles into three sectors is used to show the division of the properties into three components. The properties (Real, Nonconserved, Discrete) are represented by primary colours (for example, Red, Green, Blue), and the 'antiproperties' (Imaginary, Conserved, Continuous) by the complementary secondary colours (Cyan, Magenta, Yellow).

The choices are, of course, individually arbitrary, as is the division between properties and antiproperties. Only the overall pattern is invariable. Using the particular selection, already proposed, in Figure 1, Space becomes the identity element, in the innermost circle; the next circle then represents Charge; the next Mass; and the outermost one, Time. Figure 2 gives us an alternative representation, but it may also be used, simultaneously with Figure 1, as a representation of the *dual group*, which is a mathematical structure obtainable by switching round one of the dual pair representations, e.g. by defining space and mass as imaginary and time and charge as real, as is done in some versions of the Dirac equation.

Figure 1

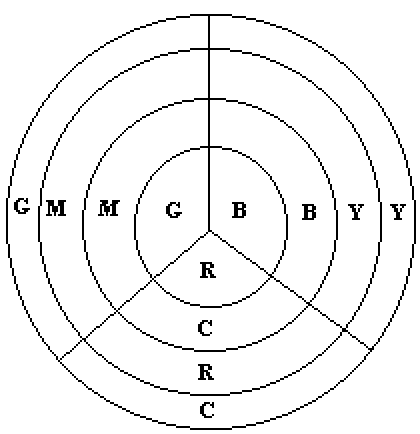
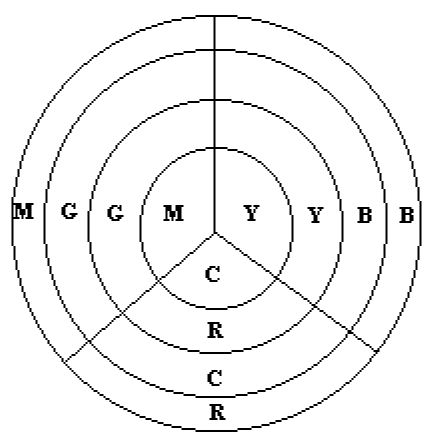


Figure 2



The nature of the fundamental parameter group is demonstrated in this representation by summing up the colour combinations in each of the circles or sectors to obtain a white totality – the representational equivalent of a total conceptual nothingness. Another way to establish this would be by representing the properties of space (real, nonconserved, divisible) by algebraic symbols, such as *x*, *y*, *z*, with the respective antiproperties (imaginary, conserved, indivisible) represented by –*x*, –*y*, –*z*:

space	*x*	*y*	*z*
time	–*x*	*y*	–*z*
mass	*x*	–*y*	–*z*
charge	–*x*	–*y*	*z*

The four 'Red' lines (shown in Figure 3 here as continuous), drawn from the origin of a set of 3-dimensional axes, then represent the four parameters, and the 'Cyan' lines (shown as dotted) those of the dual group. Figure 4 shows the same representation without the axes. The 'Red' lines are reflections of each other in two planes, while the Red *plus* Cyan lines are the reflections of each other in a single plane. The reflection of a line in three planes produces its exact dual.

Figure 3

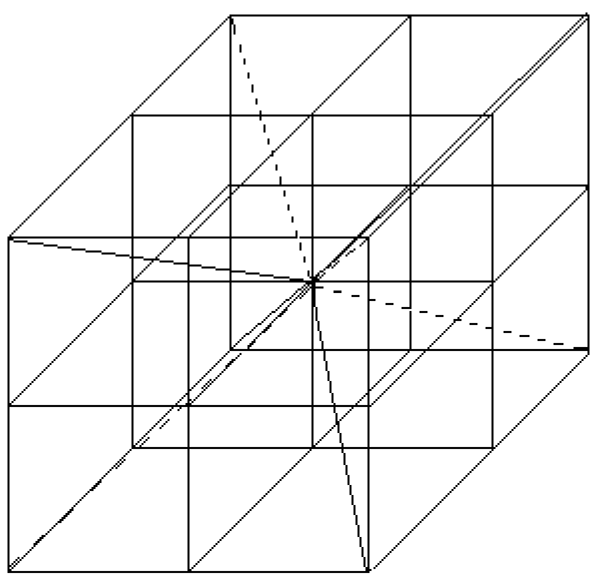
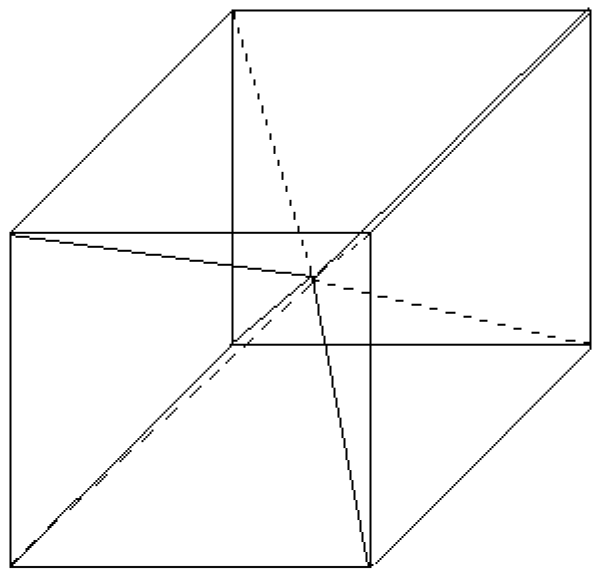
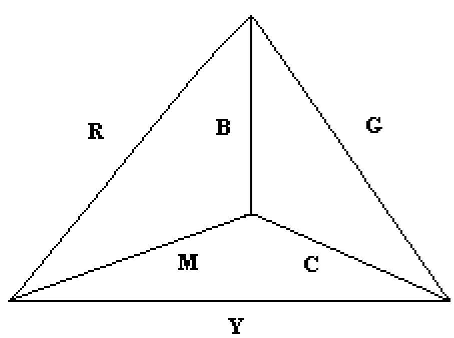


Figure 4



An alternative 3-D representation (Figure 5) situates the parameters at the vertices of a regular tetrahedron, with the six edges coloured to represent the properties and antiproperties as in Figure 1. The faces of the tetrahedron then become the members of the dual group. (The alternative representation is, of course, equally possible.)

Figure 5



4 The factor 2

The dualities in the parameter group structure may also be seen in more direct ways. For example, the number 2 appears as a factor in many physical processes, where a physical or mathematical duality is involved. Though there are many competing physical explanations for the appearance of this factor, they can all be shown to be versions of one of the more fundamental dualities. The same applies to such concepts as wave-particle duality, and the dualities between electric and magnetic forces, and between space and time in relativity. None of these is intrinsically mysterious. All have an explanation in terms of duality at the fundamental level.

Many examples show the factor 2 occurring in the duality between conjugated and nonconjugated quantities. A classic case in classical physics is that of action and reaction. This relates to the relation between potential and kinetic energies. A very general theorem in physics, the virial theorem, indicates that the time-averaged potential energy in conservative physical systems subject to either inverse-square law or constant forces (the classic ones expected for a 3-dimensional space), is equal (numerically) to twice the time-averaged kinetic energy. Effectively, the kinetic energy term corresponds to the action half of Newton's third law; the potential energy term to the action plus reaction, which preserves the total energy within the system. So the kinetic energy formula applies to a system is undergoing change, like a body of mass m escaping from a gravitational field, where

$$\frac{mv^2}{2} = \frac{GMm}{r} .$$

whereas a potential energy equation would be used for a system is in steady-state, such as a body in a classical circular gravitational orbit:

$$mv^2 = \frac{GMm}{r} .$$

The first equation, in effect, involves action only, while the second involves action plus reaction, so doubling the energy. In fact, an old proof by Newton of the mv^2 / r law for centripetal force, required to derive the second equation, was based on a doubling of the momentum through action and reaction from an imagined infinite sided polygon defining the orbit of the motion. Exactly the same momentum doubling by reflection occurs in a more directly realistic context in the derivation of the pressure-density relation for an ideal gas in steady state, $P = \rho \, \overline{c}^2 / 3$, which shows clearly the same structure as the 'potential energy' term mv^2, though divided randomly between three dimensions. An almost exactly parallel argument can be used to find the radiation pressure, $P = \rho \, c^2 / 3$, exerted by a 'photon gas'. Here, we can consider the photons as being reflected off the walls of the container in the exactly the same way as the molecules of materials or as involving the dual processes of absorption and re-emission. A related effect is that of radiation reaction, which doubles the effect when radiation is emitted by a body.

A classic case of the factor 2 is the difference between the spin angular momentum of fermions, which comes in half-integral units; and that of bosons, which comes in integral units. An indirect way of obtaining the spin ½ value of the electron is to find its moment in a magnetic field. Here, to obtain the correct answer, we use the kinetic energy equation as the ultimate origin of the factor. This is valid if we apply *at the moment we switch on the magnetic field*. Effectively, we are using the nonconserved or nonconjugated half of the duality. However, the method most often used is to calculate via the potential energy equation, and then apply relativity (in the form of the Thomas precession) to obtain the correction factor. At first sight, this suggests we are using the conserved or conjugated aspect and should double the value obtained by the first method, but, by relativistically incorporating the imaginary parameter time at the same level as the real parameter space, we are, in effect, doubling up one of the *divisors* involved in calculating the moment, and so we obtain the same result as previously.

These, of course, are indirect methods. To obtain the spin directly we use the relativistic Dirac equation. However, in this case we find that it is not relativity, but the three-dimensionality of the angular momentum term that is responsible for halving the spin, because a divisor of 2 is introduced through the anticommutativity of the vector operators (in principle, **ij** – **ji** = 2**ij**, etc.). Remarkably, then, it is possible to use *any* of the three dualities involved in the parameter group to explain the appearance of the same numerical factor in the spin term for the electron, showing that, in this respect, they have an exactly equivalent effect. This is summarized in the table below, where the words in bold type apply to the calculation procedure for the electron spin.

duality	*method*
conserved / **nonconserved**	potential energy / **kinetic energy**
real / **complex**	nonrelativistic / **relativistic**
nondimensional / **dimensional**	commutative / **anticommutative**

5 Self-duality

The most striking of all the instances of physical duality occurs in relativistic quantum mechanics, where space, time, mass and charge combine to produce an object which is *self*-dual, and so produces the desired total nothingness by acting directly on itself. (This requires more mathematical detail than the earlier discussions, but it is worth some including some discussion of it because of the profound significance of the result.) The starting-point here is the fact that the 4 parameters not only reflect the three basic dualities which underlie the mathematical structure developed in the first section, but also represent stages in the emergent algebra that it creates:

order 2	real scalar	1	mass
order 4	pseudoscalar	*i*	time
order 8	quaternions	***i***, ***j***, ***k***	charge
order 16	multivariate vectors	**i**, **j**, **k**	space

If these four mathematical structures are put together, they constitute the complex double quaternions or multivariate vector quaternions which occur at order 64, and which is the lowest order at which conjugation, complexification and dimensionalization are combined with repetition. In this algebraic structure, we have 32 possible combinations of the 8 basic units (1, *i*, ***i***, ***j***, ***k***, **i**, **j**, **k**), which become 64 when we take both + and – values. (These are equivalent to the ones given in section 1.) However, it is possible to generate the entire 64 parts from a set of 5 composite units, rather than the 8 basic ones. We take the 8 basic units:

time	space	mass	charge
i	**i j k**	1	***i j k***

Then, we superimpose each of the charge units ***k***, ***i***, ***j*** (which represent, respectively, the weak, strong and electric charges) onto one of the algebraic expressions representing by time, mass or space:

i	**i j k**	1	***i j k***
k	***i***	***j***	

to obtain the following combinations:

*i**k***	***i*i** ***i*j** ***i*k**	***j***

In creating these combined units, we are also effectively creating new physical quantities, which we can describe as energy (E), momentum (**p**) and rest mass (m), the 3-component vector term momentum being created out of the 3 central units (***i*i**, ***i*j**, ***i*k**). Combining all the units with their associated physical quantities we obtain the expression $(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\,\mathbf{p} + i\boldsymbol{j}\,m)$. If we now multiply this object by itself, we obtain zero from a well-known relativistic conservation of energy equation:

$$(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\,\mathbf{p} + i\boldsymbol{j}\,m)\,(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\,\mathbf{p} + \boldsymbol{j}\,m) = E^2 - p^2 - m^2 = 0\ .$$

In other words $(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\,\mathbf{p} + i\boldsymbol{j}\,m)$ is a square root of zero, or its *own dual*, and it can be regarded as either a classical object or, in relativistic quantum mechanics, as the state vector for a fermion as specified by the Dirac equation. In the latter case, the nonlocal connection between all the state vectors in the entire universe (described mathematically in terms of Hilbert space) provides an extension of the dualling process to infinity. Since the universe is believed to be composed entirely of fermions or fermion-antifermion combinations (bosons), the Dirac equation is, in the final analysis, the most significant way of incorporating the foundational basis for the whole of physics into a single structure, and it would appear that it is itself founded entirely on the principle of duality. Ultimately, it would seem, duality is not merely a 'component' of physics but an expression of the fundamental nature of physics itself.

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