A fundamental structure for physics

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Previous work has suggested that the Dirac or fermionic state vector in its nilpotent form is the most efficient packaging of the entire information required for the foundations of physics. All physical interactions and fundamental particles may be considered in some sense as definitions of this state. In principle, we take the four fundamental parameters time, space, mass and charge (where mass is mass-energy, the source of gravity, and charge is a 3-component quantity representing the sources of the weak, strong and electric interactions) as represented by respective pseudoscalar, multivariate vector, scalar and quaternion units:

time	space	mass	charge	
i	$\mathbf{i}\ \ \mathbf{j}\ \ \mathbf{k}$	1	$\boldsymbol{i}\ \ \boldsymbol{j}\ \ \boldsymbol{k}$	(1)

Quaternions (represented by bold italics) follow the multiplication rules:

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$$
$$\boldsymbol{ij} = -\boldsymbol{ji} = \boldsymbol{k}$$
$$\boldsymbol{jk} = -\boldsymbol{kj} = \boldsymbol{i}$$
$$\boldsymbol{ki} = -\boldsymbol{ik} = \boldsymbol{j}\ .$$

Multivariate vectors are equivalent to *complexified* quaternions (multiplied by pseudoscalar i), we obtain, for the complex terms, which would have multiplication rules

$$(i\boldsymbol{i})^2 = (i\boldsymbol{j})^2 = (i\boldsymbol{k})^2 = (i\boldsymbol{i})(i\boldsymbol{j})(i\boldsymbol{k}) = 1$$
$$(i\boldsymbol{i})(i\boldsymbol{j}) = -(i\boldsymbol{j})(i\boldsymbol{i}) = i\boldsymbol{k}$$
$$(i\boldsymbol{j})(i\boldsymbol{k}) = -(i\boldsymbol{k})(i\boldsymbol{j}) = i\boldsymbol{i}$$
$$(i\boldsymbol{k})(i\boldsymbol{i}) = -(i\boldsymbol{i})(i\boldsymbol{k}) = (i\boldsymbol{j})\ .$$

It is convenient, however, to use a separate symbolism (here represented by bold characters), so that:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = 1$$
$$\mathbf{ij} = -\mathbf{ji} = i\mathbf{k}$$
$$\mathbf{jk} = -\mathbf{kj} = i\mathbf{i}$$
$$\mathbf{ki} = -\mathbf{ik} = i\mathbf{j}\ .$$

We may recognize the multiplication rules here as being identical to those for Pauli matrices, and Hestenes and others (*Space-time Algebras*, Gordon and Breach, New York, 1966) have demonstrated that, adopting such rules for the vectors in quantum mechanics, allows us to explain the origin of the otherwise mysterious property of spin.

Taking into account every conceivable combination of the units, the complete algebra associated with these eight symbols is a 32-part (or group 64, if we incluse + and

– signs) quaternion-multivariate vector (or complex double quaternion) algebra. However, the *most efficient* way of generating this algebra is not from the eight basic units, but from five unit combinations created by taking the three units of either of the 3-dimensional parameters (space or charge) onto the other five. The most convenient choice is charge, in which case the composite units become

$$i\boldsymbol{k} \qquad \mathbf{i}\boldsymbol{i} \;\; \mathbf{j}\boldsymbol{i} \;\; \mathbf{k}\boldsymbol{i} \qquad 1\boldsymbol{j} \tag{2}$$

which now have new physical properties, created by their combined natures, and which we may recognize as as those of (quantized) energy, momentum and rest mass:

$$E \qquad \mathbf{p} \qquad m \tag{3}$$

In addition to creating these new quantities out of space, time, mass and charge, this ‘packaging’ process has the effect of breaking the symmetry between the 3 charge units, which now take on the respective pseudoscalar, vector and scalar characterictics that we recognize as those of and also those of weak charge, strong charge and electric charge:

$$w \qquad s \qquad e \tag{4}$$

The packaging automatically requires the connections between space and time and energy and momentum that we describe as ‘relativistic’ (with the added feature of explicit causality required by the fifth term); the discrete and conserved nature of the charge units further imposes the condition that we descibe as quantization. In the nilpotent formalism, the packaging or compactification is the *source* of both these fundamental conditions, not an effect of them. It also has the remarkable effect of changing the completely symmetric but less compact representation (1) into the asymmetric but more compact representation (2). It has remarkable parallels with the Higgs mechanism, where the ground state is not the most symmetric one, and is indeed the ultimate source of that physical phenomenon. Physically, the asymmetry appears to manifest itself most obviously in the pseudoscalar term (the first one in (1), (2), (3) and (4)): energy is always positive; time is irreversible; the weak interaction breaks charge conjugation symmetry. It is significant that 5-fold patterns are the ones in which perfect symmetry first breaks down in nature (for example, in crystal structures, tilings, or Platonic solids); and it is equally significant that the 5-fold Dirac or fermionic state, which emerges out of (2) and (3) can only occur in a unique and unrepeatable form (Pauli exclusion), like the patterns in the 5-fold Penrose tiling.

The structure of the Dirac or fermionic state emerges naturally when we realize that the mathematical units represented in (2) are exactly equivalent to the five gamma matrices used in the conventional Dirac equation:

$$(\gamma^{\mu}\partial_{\mu} + im)\ \psi = 0\ , \tag{5}$$

which, in component form, becomes:

$$\left(\gamma^0 \frac{\partial}{\partial t} + \gamma^1 \frac{\partial}{\partial x} + \gamma^2 \frac{\partial}{\partial y} + \gamma^3 \frac{\partial}{\partial z} + im\right) \psi = 0\ . \tag{6}$$

The conventional 4 × 4 matrices are inconvenient because they cause fragmentation of the equation, mixing up energy, momentum and mass term. They also take up too much logical space, requiring 16 pieces of information for one operation. In addition, the fifth matrix, though required to complete the algebra, does not appear in (5) and (6), although the operators in the equations have five terms.

Substition of algebraic operators of the kind represented in (2) for the gamma matrices allows us to overcome all these problems, though the nilpotent nature of the formalism is only revealed after a few further algebraic transformations, and identification of the terms in the 4-spinor ψ as being generated out of the 4 combinations of $\pm \boldsymbol{k}E$ (fermion / antifermion) and $\pm i\mathbf{i}\mathbf{p}$ (spin up / down). In the case of a free particle, the result is a fermionic state, with amplitude $(\pm \boldsymbol{k}E \pm i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)$. Here the terms have been multiplied throughout by i (to equate with existing conventions), and the four possible sign combinations are taken (again by convention) to be components in a row or column vector. The expression has the remarkable property of being a nilpotent, or square root of zero:

$$(\pm \boldsymbol{k}E \pm i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)\,(\pm \boldsymbol{k}E \pm i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m) = -E^2 + p^2 + m^2 = 0\ ,$$

which is, incidentally, an exact expression of Pauli exclusion. For a free particle, also, we define a 4-spinor, not *matrix*, operator $(\pm i\boldsymbol{k}\partial/\partial t \pm \boldsymbol{i}\nabla + i\boldsymbol{j}\,m)$, which, by operating on the phase $exp\,(i(Et - \mathbf{p.r}))$, produces the eigenvalue $(\pm \boldsymbol{k}E \pm i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)$.

However, it is more convenient to define $(\pm \boldsymbol{k}E \pm i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)$ as the operator, which we can then apply more generally to fermions of any description whether bound or free. That is, the E and $\mathbf{p}$ terms may be considered general expressions, which may represent covariant derivatives or incorporate field terms of any kind. We now define the fermionic state to be simply an operator of this form, which is required to *find a phase* to which it may be applied in such a way that the resulting expression for the amplitude is a nilpotent. In this way, it is no longer necessary to use the Dirac equation directly, or to use a wavefunction to express an entire physical state. We need only specify an operator of the form $(\pm \boldsymbol{k}E \pm i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)$, and the rest follows automatically. This is what we mean by ‘solving’ the ‘equation’ for a particular field term or set of field terms. It has been shown in previous work that, for a single point particle state (to which spherical symmetry would necessarily apply), there are only three potentials with distance dependence which give nilpotent solutions – inverse linear; inverse linear plus linear; and inverse linear plus any polynomial term(s) other than linear – and that these have the respective characteristics of the electric, strong and weak interactions, and that they correspond to respective scalar, vector and pseudoscalar terms in the nilpotent operator.

The four components of the nilpotent spinor may be shown to be equivalent to creation (or annihilation) operators for fermion / antifermion, with spins up / down. We can also immediately recognize the state vectors for

Fermion	$(\pm \boldsymbol{k}E \pm i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)$
Fermion with reversed spin	$(\pm \boldsymbol{k}E \mp i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)$
Antifermion	$(\mp \boldsymbol{k}E \pm i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)$
Antifermion with reversed spin	$(\mp \boldsymbol{k}E \mp i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)$

Spin ½ of fermions and one-handed helicity can be derived in the conventional way (relative helicity is determined by the relative signs of E and $\mathbf{p}$).

Bosons are effectively interaction vertices for fermionic states and we can easily construct state vectors for spin 1 bosons, spin 0 bosons and the kind of combinations that produce Bose-Einstein condensates as respectively scalar products of fermion / antifermion with the same helicity; fermion / antifermion with opposite helicity; and fermion / fermion with opposite helicity.

Spin 1 boson	$(\pm \boldsymbol{k}E \pm i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)\ (\mp \boldsymbol{k}E \pm i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)$
Spin 0 boson	$(\pm \boldsymbol{k}E \pm i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)\ (\mp \boldsymbol{k}E \mp i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)$
Bose-Einstein condensate	$(\pm \boldsymbol{k}E \pm i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)\ (\pm \boldsymbol{k}E \mp i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)$

Spin 1 bosons, significantly, may be massless but massless spin 0 bosons would disappear, with their scalar product becoming $E^2 - p^2 = 0$. A degenerate massless vacuum state cannot therefore exist without creating a massive boson.

Bosons are structured as unified states, with E, $\mathbf{p}$ and m values common to the fermionic and antifermionic parts, and we can, in fact, postulate that the signature of a completely interacting dynamical theory of composite particles is that the E, $\mathbf{p}$ and m values have meaning only in the context of the *entire state*. It is also evident that nilpotent operators are naturally supersymmetric, with supersymmetry operators:

Boson to fermion:	$Q = (\pm \boldsymbol{k}E \pm i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)$
Fermion to boson:	$Q^\dagger = (\mp \boldsymbol{k}E \pm i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)$

So we can use Q and $Q^\dagger$ to convert bosons to fermions / antifermions and vice versa. The supersymmetry is exact. Such exact supersymmetry suggests particles are their own supersymmetric partners.

So it is possible to have a nonzero state vector if we use the *vector* properties of $\mathbf{p}$ and the arbitrary nature of its sign (+ or –). A state vector of the form, privileging the $\mathbf{p}$ components:

$$(\boldsymbol{k}E \pm i\boldsymbol{i}\ \mathbf{i}p_X + i\boldsymbol{j}\ m)\ (\boldsymbol{k}E \pm i\boldsymbol{i}\ \mathbf{j}p_Y + i\boldsymbol{j}\ m)\ (\boldsymbol{k}E \pm i\boldsymbol{i}\ \mathbf{k}p_Z + i\boldsymbol{j}\ m)$$

(where, for convenience, we write only the first line of the 4-spinor) will have six possible phases with $\mathbf{p} = \pm\ \mathbf{i}p_X$; $\mathbf{p} = \pm\ \mathbf{j}p_Y$; $\mathbf{p} = \pm\ \mathbf{k}p_Z$, exactly as required for a baryon, with $SU(3)$ symmetry automatic, and spin 1 gluon states of the form $(\boldsymbol{k}E \pm i\boldsymbol{i}\ \mathbf{i}p_X)\ (-\boldsymbol{k}E \pm i\boldsymbol{i}\ \mathbf{j}p_Y)$. Such a state would necessarily have nonzero mass because the complete symmetry requires both + and – signs for $\mathbf{p}$, and the application of a gluon would always change this sign.

For many further results and developments, see *PIRT Proceedings* and P. Rowlands, arXiv:physics/0106054; quant-ph/0301071; *AIP Conference Proceedings*, 78, 102-115, 2004. P. Rowlands and J. P. Cullerne, arXiv:quant-ph/00010094; 0103036; 0109069; and *Nuclear Physics* A 684, 713-5, 2001. P. Rowlands and B. Diaz, arXiv:cs.OH/0209026.